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Wave-induced longitudinal-vortex instability in shear flows

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The instability of two-dimensional periodic flows to spanwise-periodic 'longitudinal-vortex' modes is examined. The undisturbed state comprises a parallel shear flow and a two-dimensional $O(\epsilon)$ wave field as encountered in, say, water-wave or hydrodynamic-stability theories.

When the mean shear is weak, of order ϵ^2, the present theory reduces to that of Craik (1977) and Leibovich (1977*b*, 1980). For stronger but still weak shear, of order ϵ, it is established that the Craik–Leibovich instability mechanism is essentially unchanged, apart from scaling factors.

For strong $O(1)$ shear flows, the governing equations are derived by using, in part, a generalized Lagrangian-mean formulation. The resultant eigenvalue problem for the longitudinal-vortex instability is then more complex, but simplifies in the case of small spanwise spacing of the vortices, in the inviscid limit. An example is given of flows that exhibit instability in this limiting case. Such instability seems likely to occur for a wide class of periodic shear flows. Complementary physical interpretations of the instability mechanism are discussed.

1. Introduction

Longitudinal vortices are prominent features of shear-flow turbulence. Also, the development of such vortices from initially small spanwise irregularities in unstable boundary layers has been revealed by the careful experiments of Klebanoff, Tidstrom & Sargent (1962). In these and in later experiments, notably those of Nishioka, Iida & Ichikawa (1975) on plane Poiseuille flow and Saric & Reynolds (1981 private communication) on Blasius flow, the wave field also develops a marked 'peak-and-valley' structure, which is clear evidence of the growth of oblique-wave modes (for a recent discussion see Craik 1980). Associated theoretical work by Benney & Lin (1960), Benney (1964), Antar & Collins (1975) and others calculates the longitudinal-vortex flow that is driven by the interaction of selected two-dimensional and oblique-wave modes. In such cases, the longitudinal vorticity at first grows linearly in time t (or distance x) for constant-amplitude waves, while the spanwise-periodic component of downstream velocity starts to grow as t^2 (or x^2).

In practice, a preferred spanwise spacing may develop spontaneously, even when the downstream-propagating waves are initially two-dimensional (Anders & Blackwelder 1980). This may be due to selective amplification of certain oblique-wave modes either by linear or nonlinear processes, the longitudinal vortices being forced by the subsequent nonlinear interaction of the two- and three-dimensional wave modes. Alternatively, it is possible that the two-dimensional periodic flow is unstable to disturbances of longitudinal-vortex form: in which case, such structures might grow exponentially in time t until a finite-amplitude equilibrium state is reached. In the latter event, it is the nonlinear coupling between the fundamental two-dimensional waves and the spanwise-periodic flow that would generate oblique-wave modes – a reversal of the former evolutionary process.

The latter possibility has received little theoretical attention (see Herbert & Morkovin 1980), though it is no less rational than the Benney–Lin mechanism. Nayfeh (1981) has examined the effect of constant-amplitude vortices on the growth rates of oblique waves. An exponential growth of the vortices themselves has not previously been proposed for strong shear flows. But an instability of rather similar form has been studied in the context of *weak* mean shear flows by Craik (1977, 1982*a*), Leibovich (1977*b*, 1980) and Leibovich & Paolucci (1980, 1981). This work was developed as an explanation of the phenomenon of 'Langmuir circulations' in lakes and oceans. In it, the basic state consists of two-dimensional $O(\epsilon)$ waves and an $O(\epsilon^2)$ Eulerian mean shear flow $\bar{u}(z)$. The latter flow may be maintained by a wind stress or it may result from wave dissipation by viscosity: in either case, the so-called Craik–Leibovich instability mechanism will normally operate to generate spanwise-periodic flows. In the absence of viscous dissipation, the exponential growth rate of the Craik–Leibovich instability is $O(\epsilon^2)$. Accordingly, with sufficiently small wave slopes ϵ, the instability can be suppressed by viscous damping; but, for typical water waves, this occurs only at extremely small amplitudes, and instability is normally to be expected.

The mathematical structure of the ultimate governing equations of the Craik–Leibovich theory resembles that for onset of Bénard convection or of Taylor vortices between concentric rotating cylinders. The physical mechanism has been cogently explained as a kinematical process in which vortex-line deformation by the Stokes-drift gradient of the wave field combines with vertical advection of the mean Eulerian flow by the vortex motion (see Craik 1977; Leibovich 1977*b*). The present work began as an attempt to discover whether the Craik–Leibovich instability might continue to operate for stronger mean shear flows than those originally envisaged. This was readily found to be so for larger, but still small, shears $O(\epsilon)$, with only minor modifications of the theory. For such weak shears, an alternative, dynamical explanation of the instability is given, in which it is identified rather closely with Taylor–Görtler instability of curved flows, but averaged in the downstream direction, where the local curvature corresponds to that of the undulating streamlines.

For strong $O(1)$ shear flows, on the other hand, a completely new theory has had to be constructed, though the seminal idea of the Craik–Leibovich work remains within it. The analysis is largely based on the generalized Lagrangian-mean (GLM) formulation of Andrews & McIntyre (1978*a*, *b*; hereinafter referred to as I and II) as outlined in the preceding paper Craik (1982*b*) for strong shear flows. This formulation quickly yields the solution for $O(\epsilon^2)$ and $O(\epsilon)$ shear flows, the GLM equations then reducing directly to their Eulerian counterparts in much the same way as shown by Leibovich (1980). However, with $O(1)$ shear flows, the back-effect of the mean-flow modification upon the wave field must be explicitly calculated, and the resultant stability problem for the longitudinal vortices is much more complex.

The inviscid stability problem is specified in a form suitable for future numerical computation of particular cases. In addition, an approximation for large spanwise wavenumbers is developed, for which the problem simplifies sufficiently to allow analytical solution. An example is given that demonstrates the existence of longitudinal-vortex instability in strong shear flows. It is likely that such instability will occur in a wide class of shear flows perturbed by two-dimensional waves.

2. General formulation

2.1. *Governing equations*

We consider the GLM equations for cases where all mean quantities are independent of the streamwise coordinate $x = x_1$. (A possible exception is mean pressure π, which may have an x_1 independent gradient $\partial\pi/\partial x_1$ for viscous channel flows). For constant-density fluid in a non-rotating reference frame, these equations are, from I (3.8),

$$\bar{D}^{\mathrm{L}}(\bar{u}_1^{\mathrm{L}} - p_1) + \pi_{,1} = -X_1, \tag{2.1a}$$

$$\bar{D}^{\mathrm{L}}(\bar{u}_2^{\mathrm{L}} - p_2) + \bar{u}_{k,2}^{\mathrm{L}}(\bar{u}_k^{\mathrm{L}} - p_k) + \pi_{,2} = -X_2, \tag{2.1b}$$

$$\bar{D}^{\mathrm{L}}(\bar{u}_3^{\mathrm{L}} - p_3) + \bar{u}_{k,3}^{\mathrm{L}}(\bar{u}_k^{\mathrm{L}} - p_k) + \pi_{,3} = -X_3. \tag{2.1c}$$

The notation is identical with that of the preceding paper (Craik 1982*b*, hereinafter referred to as C). Numerical suffixes relate to components along Cartesian axes $(x_1, x_2, x_3) \equiv (x, y, z)$, the comma denotes partial differentiation, and summation over repeated indices is implied. The operator $\bar{D}^{\mathrm{L}}$ is defined as $\bar{D}^{\mathrm{L}} \equiv \partial/\partial t + \bar{u}_j^{\mathrm{L}}\partial/\partial x_j$; $\bar{u}_i^{\mathrm{L}}$ and p_i are Lagrangian-mean velocity and pseudomomentum respectively, and the X_i denote dissipative terms. For simplicity in what follows, we consider only inviscid fluids for which the X_i are identically zero; but such terms might be retained (see Leibovich 1980; Grimshaw 1982).

2.2. *Weak $O(\epsilon^2)$ mean flow*

We first briefly review the situation discussed in detail by Leibovich (1980), which relates to the Craik–Leibovich instability mechanism with weak $O(\epsilon^2)$ mean flows. The cases of $O(\epsilon)$ and $O(1)$ mean flows are then examined.

An initially two-dimensional wave field of, say, surface gravity waves for definiteness, has wave slopes characterized by the small parameter ϵ. This gives rise to a unidirectional $O(\epsilon^2)$ Stokes drift $[\bar{u}_1^{\mathrm{S}}(z), 0, 0]$. Also present is a *weak* $O(\epsilon^2)$ Eulerian-mean shear flow $[\bar{u}(z), 0, 0]$. We now envisage a small spanwise-periodic perturbation with Eulerian velocity components of the form

$$(\tilde{u}, \tilde{v}, \tilde{w}) = \epsilon^2\delta\,\mathrm{Re}\,\{e^{\sigma t}\,e^{ily}\,[\hat{u}(z), \hat{v}(z), \hat{w}(z)]\}, \tag{2.2a}$$

where

$$il\hat{v} + \hat{w}_{,3} = 0 \tag{2.2b}$$

by continuity. δ is a second small parameter, which measures the strength of this motion relative to the primary $O(\epsilon^2)$ shear flow. This δ is assumed sufficiently small that linearization with respect to δ yields a good approximation to the equations governing the spanwise-periodic disturbance. The growth or decay rate σ of the spanwise vortices should not be confused with the growth rate of the waves (for which σ was also used in C). Throughout the present paper, the $O(\epsilon)$ waves are assumed to be of constant amplitude.

In such cases, the pseudomomentum per unit mass of the unperturbed state equals the Stokes drift $[\bar{u}_1^{\mathrm{S}}, 0, 0]$ (cf. C, §3; Leibovich 1980) with an error that is $O(\epsilon^4)$. Accordingly, at $O(\epsilon^2)$, $\bar{u}_i^{\mathrm{L}} - p_i$ is just the Eulerian velocity distribution $[\bar{u}(z), 0, 0]$. Also, the distortion of the primary $O(\epsilon)$ wave field by the $O(\epsilon^2\delta)$ spanwise-periodic currents yields wave components of order $O(\epsilon^3\delta)$ and hence Stokes-drift and pseudomomentum perturbations of order $O(\epsilon^4\delta)$. Since such perturbations may be ignored in comparison with the $O(\epsilon^2\delta)$ Eulerian velocity perturbations, the whole $O(\epsilon^2\delta)$ contribution to $\bar{u}_i^{\mathrm{L}} - p_i$ still comes from the Eulerian velocity field. It also follows that the contributions to $\bar{u}_i^{\mathrm{L}}$ at $O(\epsilon^2)$ and $O(\epsilon^2\delta)$ are $[\bar{u}(z) + \bar{u}_1^{\mathrm{S}}(z), 0, 0]$ and $[\tilde{u}, \tilde{v}, \tilde{w}]$ respectively.

For inviscid flow, (2.1) and (2.2) then reduce to

$$\sigma\hat{u} = -\hat{w}\bar{u}', \tag{2.3a}$$

$$\sigma(\hat{v}_{,3} - il\hat{w}) = il\hat{u}\bar{u}_1^{S\prime}, \tag{2.3b}$$

$$il\hat{v} + \hat{w}_{,3} = 0, \tag{2.3c}$$

where the prime denotes d/dz, and the pressure π has been eliminated by cross-differentiation. Note that the right-hand sides of (2.3*a*, *b*) are $O(\epsilon^4\delta)$, which implies that σ must be $O(\epsilon^2)$. These equations yield

$$\hat{w}_{,33} + l^2[(\bar{u}_1^{S\prime}\bar{u}'/\sigma^2) - 1]\hat{w} = 0 \tag{2.4}$$

in terms of $\hat{w}$ only, which, together with appropriate boundary conditions (typically $\hat{w} = 0$ at levels $z = z_1$ and z_2), defines an eigenvalue problem for $\sigma = \sigma(l)$. For these boundary conditions, σ normally has real roots, indicating instability, when $\bar{u}_1^{S\prime}\bar{u}'$ takes positive values in some part of the flow domain. Several examples are given by Craik (1977), who also considers the viscous case; Leibovich (1977*b*) discusses the inviscid eigenvalue problem (2.4), and its equivalent for stratified flows, in greater generality; while Leibovich & Paolucci (1981) determine stability criteria for time-dependent mean flows $\bar{u}(z,t)$.

2.3. *Weak $O(\epsilon)$ mean flow*

We turn now to cases in which the basic Eulerian shear flow $\bar{u}(z)$ is $O(\epsilon)$; that is, of comparable magnitude to the wave orbital velocities. The primary Stokes drift and pseudomomentum remain unchanged at $O(\epsilon^2)$, and these are nearly equal, with an error of $O(\epsilon^3)$. We again envisage spanwise-periodic perturbations as in (2.2), but with $\epsilon^2\delta$ now provisionally replaced by $\epsilon\delta$. Such perturbations induce spanwise-varying distortions of the wave field that are $O(\epsilon^2\delta)$ and consequent Stokes drift and pseudomomentum perturbations $O(\epsilon^3\delta)$. The latter are again negligible in comparison with the $O(\epsilon\delta)$ Eulerian-velocity perturbations. Proceeding as above, equations identical with (2.3*a*–*c*) are again obtained; but now the right-hand sides of (2.3*a*) and (2.3*b*) are respectively $O(\epsilon|\hat{w}|)$ and $O(\epsilon^2|\hat{u}|)$. These imply the scalings

$$\left|\frac{\hat{w}}{\hat{u}}\right|, \quad \left|\frac{\hat{v}}{\hat{u}}\right| \sim O(\epsilon^{\frac{1}{2}}); \quad \sigma \sim O(\epsilon^{\frac{3}{2}}).$$

Accordingly, the counterpart of (2.2*a*, *b*) should be rechosen as

$$(\tilde{u}, \tilde{v}, \tilde{w}) = \epsilon\delta\,\mathrm{Re}\{e^{\sigma t}\,e^{ily}[\hat{u}(z), \epsilon^{\frac{1}{2}}\hat{v}(z), \epsilon^{\frac{1}{2}}\hat{w}(z)]\},$$

with $\sigma \equiv \epsilon^{\frac{3}{2}}\sigma_1$.

On noting that the Stokes drift and pseudomomentum distortions at $O(\epsilon^3\delta)$ remain small compared with the rescaled $\tilde{v}$- and $\tilde{w}$-components, which are $O(\epsilon^{\frac{3}{2}}\delta)$, one readily recovers (2.4) for this case also. The same instability mechanism therefore operates in the presence of such stronger $O(\epsilon)$ mean shear flows.

2.4. *Strong shear flow*

Thirdly, we consider cases of strong $O(1)$ Eulerian-mean shear flows of the kind discussed in C. The waves may be constant-amplitude free-surface waves or waves on any compliant boundary in a strong shear flow. Alternatively, they may be neutrally stable modes, contained between plane boundaries at z_1 and z_2. We therefore have an $O(1)$ primary flow $\bar{\mathbf{u}}^{\mathrm{L}} = [\bar{u}(z), 0, 0]$, an $O(\epsilon)$ wavelike motion, as described in

C(3.2), and $O(\epsilon^2)$ Stokes drift and pseudomomentum as in C(3.3) and C(3.4). Since the waves are here of constant amplitude, we set $c_i = 0$ in all these results.

We first postulate a spanwise-periodic perturbation of the form

$$(\tilde{u}, \tilde{v}, \tilde{w}) = \delta \,\mathrm{Re}\{e^{\sigma t} e^{ily}[\hat{u}(z), \hat{v}(z), \hat{w}(z)]\}. \tag{2.5}$$

Induced perturbations in the wave field are of strength $O(\epsilon\delta)$, giving Stokes-drift and pseudomomentum perturbations of $O(\epsilon^2\delta)$. The Lagrangian-mean velocity at $O(\delta)$ is therefore identical with (2.5), and the $O(\delta)$ equations are just

$$\sigma\hat{u} = -\hat{w}\bar{u}', \tag{2.6a}$$

$$\sigma(\hat{v}_{,3} - il\hat{w}) = 0, \tag{2.6b}$$

$$il\hat{v} + \hat{w}_{,3} = 0, \tag{2.6c}$$

on eliminating π. These lead to

$$\sigma(\hat{w}_{,33} - l^2\hat{w}) = 0, \quad \hat{u} = -\bar{u}'\sigma^{-1}\hat{w},$$

which has no non-zero solution for boundary conditions of the form $\hat{w} = 0$ at $z = z_1$ and z_2, and which therefore cannot lead to instability.

A rescaling of the perturbation is necessary, to

$$(\tilde{u}, \tilde{v}, \tilde{w}) = \delta \,\mathrm{Re}\{e^{\sigma t} e^{ily}[\hat{u}(z), \epsilon\hat{v}(z), \epsilon\hat{w}(z)]\}, \tag{2.7}$$

where $\sigma \equiv \epsilon\sigma_1$ is an $O(\epsilon)$ growth rate, and the y- and z-velocity components are weaker by a factor ϵ than the downstream component. Clearly, the $O(\epsilon^2\delta)$ Stokes-drift and pseudomomentum perturbations remain negligible compared with the rescaled $O(\epsilon\delta)$ Eulerian velocity components; but this does not now mean that these perturbations may be disregarded. From (2.1 *b*, *c*) one finds at $O(\epsilon^2\delta)$ that

$$\epsilon\delta\sigma(\hat{v}_{,3} - il\hat{w}) = \delta il\hat{u}p^0_{1,3} - il\bar{u}'\tilde{p}_1,$$

where $p^0_{1,3}$ denotes the z-derivative of the $O(\epsilon^2)$ pseudomomentum C(3.4 *a*), and $\mathrm{Re}\{e^{\sigma t + ily}\tilde{p}_1\}$ is the $O(\epsilon^2\delta)$ spanwise-periodic perturbation of p_1. Similarly, at $O(\epsilon\delta)$, (2.1 *a*) yields

$$\delta\sigma\hat{u} = -\epsilon\delta\hat{w}\bar{u}'. \tag{2.8}$$

On writing $p^0_1 = \epsilon^2 P^0_1$ and $\tilde{p}_1 = \epsilon^2\delta\hat{P}_1$, the final $O(\epsilon\delta)$ equation for $\hat{w}$ takes the form

$$\hat{w}_{,33} + l^2\left[\frac{P^0_{1,3}\bar{u}'}{\sigma_1^2} - 1\right]\hat{w} = \frac{-l^2\bar{u}'}{\sigma_1}\hat{P}_1. \tag{2.9}$$

The left-hand side of this equation is similar to that of (2.4), with the $O(\epsilon^2)$ Stokes drift $\bar{u}^S_1$ replaced by the pseudomomentum p^0_1; but now the right-hand side depends upon the distortion of the wave field through $\hat{P}_1$. The GLM equations (2.1) provide no direct means of evaluating $\hat{P}_1$, and a separate examination of the wave field is necessary.

Since it is only the $O(\epsilon^2\delta)$ contribution to p_1 that is required, the influence on the waves of the $O(\epsilon\delta)$ velocity components $\tilde{v}$ and $\tilde{w}$ may be ignored. That is to say, the significant part of the distortion of the wave field is due to the $O(\delta)$ spanwise-periodic x-velocity $\tilde{u}$. Accordingly, in calculating $\hat{P}_1$ one need only consider the linearized theory of wave motion in the presence of a mean flow $\bar{u} + \tilde{u}$, in which $\tilde{u}$ represents weak spanwise variations. Whether the term in $\hat{P}_1$ acts to reinforce or to inhibit the instability remains to be seen.

2.5. *Three-dimensional waves*

Finally in this section, we consider the generation of spanwise-periodic flows by wave fields that contain $O(\epsilon)$ oblique-wave components. This is the situation first envisaged by Benney & Lin (1960) for boundary layers, and which formed the basis of the early work of Craik (1970), Leibovich & Ulrich (1972) and Craik & Leibovich (1976) on Langmuir circulations. Now, the initial state consists of a mean Eulerian velocity $\bar{u}(z)$ and a wave field with characteristic slopes of order $O(\epsilon)$, comprising both two-dimensional (in x_1 and x_3) and oblique wave components. The associated Stokes drift $\bar{\mathbf{u}}^{\mathrm{S}}$ and pseudomomentum are then spanwise-periodic.

When the Eulerian mean shear $\bar{u}'$ is weak, of order ϵ^2, Leibovich (1980) has shown that (2.1) reduce to the equations originally derived by Craik & Leibovich (1976), which are expressed in terms of Eulerian velocity components and the *prescribed* Stokes drift of the wave field (the results (2.3) are a special case of these). A spanwise-periodic Stokes drift then distorts the initially uniform spanwise vorticity field $\bar{u}'(z)$ to generate spanwise-periodic longitudinal (x_1) vorticity that grows linearly with time t; while the associated spanwise-periodic downstream velocity $\tilde{u}$ initially grows as t^2 (cf. Leibovich & Ulrich 1972). Associated viscous problems have been treated by Craik & Leibovich (1976), Leibovich (1977*a*) and Leibovich & Radhakrishnan (1977).

When the Eulerian-velocity gradient $\bar{u}'(z)$ is $O(1)$ and the pseudomomentum is spanwise-periodic, the $O(\epsilon^2)$ equations corresponding to (2.1) reduce to

$$\frac{\partial}{\partial t}(\bar{u}_1^{\mathrm{L}}-p_1) = -\bar{u}_3^{\mathrm{L}}\bar{u}', \tag{2.10a}$$

$$\frac{\partial}{\partial t}[(\bar{u}_2^{\mathrm{L}}-p_2)_{,3}-(\bar{u}_3^{\mathrm{L}}-p_3)_{,2}] = -p_{1,2}\bar{u}'. \tag{2.10b}$$

Now, if the x_2 and x_3 components of Stokes drift and pseudomomentum should happen to be zero, or equal to one another, at this order, the term in square brackets is just the mean longitudinal vorticity. More generally, this term is the x_1 component of a vector field that is associated with the vorticity field by a simple mapping (see I, §7). This equation therefore describes the rate of creation of x_1 vorticity by the tilting of the spanwise-vorticity field $\bar{u}'$ by the spanwise-periodic pseudomomentum gradient $p_{1,2}$. Initially, such vorticity would grow linearly with time t, with consequent growth of the spanwise-varying part of $\bar{u}_1^{\mathrm{L}}$ proportional to t^2. Such situations are described in the inviscid analyses of, for example, Benney (1964) and Craik (1970), who employed Eulerian variables throughout.

However, the GLM equations are again insufficient to allow us to solve these problems, because distortions of the Stokes drift and pseudomomentum develop at the same rate and with the same strength as those of $\bar{\mathbf{u}}^{\mathrm{L}}$. The required evolution equations for $\mathbf{p}$ and $\bar{\mathbf{u}}^{\mathrm{S}}$ must include the influence of the developing mean flow upon the *fluctuating* part of the motion. In this sense, the GLM equations (2.1) are incomplete, for they describe the effect of fluctuations upon the mean state, but not vice versa (cf. McIntyre 1980).

3. The eigenvalue problem for strong shear flows

We here formulate the inviscid stability problem for spanwise-periodic disturbances in the presence of a strong mean flow $\bar{u}(z)$ and $O(\epsilon)$ two-dimensional straight-crested periodic waves that propagate in the flow direction. The governing equation for spanwise-periodic mean flows of the form (2.7) is given in (2.9). In this, $P^0_{1,3}$ is known in terms of the primary wave field, since

$$P^0_1 = -\frac{\bar{u}}{2}\left\{\left|\left(\frac{\phi}{\bar{u}}\right)'\right|^2 + \alpha^2\left|\frac{\phi}{\bar{u}}\right|^2\right\}, \tag{3.1}$$

from C(3.4a).

The pseudomomentum component $\hat{P}_1$ must be recovered from

$$p_1 = -\overline{\xi_{j,1}u^l_j} = -\overline{\xi_{j,1}u'_j} - \bar{u}^{\mathrm{E}}_{1,k}\overline{\xi_{1,1}\xi_k}, \tag{3.2}$$

where $\bar{u}^{\mathrm{E}}_1 = \bar{u}(z) + \tilde{u}(y,z,t)$ (we recall that it is sufficient to consider only the $O(\delta)$ downstream component of the mean flow). The fluctuations $\hat{u}_j$ and ξ_j have $O(\epsilon)$ contributions from the primary wave field and $O(\epsilon\delta)$ terms that are proportional to $\exp[i(\alpha x \pm ly) + \epsilon\sigma_1 t]$. After reduction, $\hat{P}_1$ may be expressed in the form

$$\hat{P}_1 = \mathscr{A}(z)\,\hat{u} + \mathscr{B}(z)\,\hat{u}_{,3} + \mathrm{Re}\,\{\mathscr{C}(z)\,\hat{\phi} + \mathscr{D}(z)\,\hat{\phi}_{,3}\} \tag{3.3}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ are functions independent of σ_1, and $\hat{\phi}(z)$ relates to the $O(\epsilon\delta)$ spanwise-periodic wave-field modification. The derivative of this expression for $\hat{P}_1$ is given in §4. It is also shown in §4 that the wave-field modification satisfies the inhomogeneous equation

$$\bar{u}\left[\frac{d^2}{dz^2} - (\alpha^2 + l^2)\right]\hat{\phi} - \bar{u}''\hat{\phi} = -\hat{u}\left[\frac{d^2}{dz^2} - (\alpha^2 + l^2)\right]\phi + \hat{u}_{,zz}\phi. \tag{3.4}$$

Here $\phi(z)$ and α denote the eigenfunction and wavenumber of the primary wave field, which satisfy Rayleigh's equation

$$\bar{u}(\phi'' - \alpha^2\phi) - \bar{u}''\phi = 0. \tag{3.5}$$

Finally, $\hat{u}$ and $\hat{w}$ are related by (2.8):

$$\sigma_1\hat{u} = -\bar{u}'\hat{w}. \tag{3.6}$$

We note that the eigenvalue σ_1 occurs explicitly only in (2.9) and (3.6).

The coupled system (2.9), (3.3), (3.4) and (3.6), together with appropriate homogeneous boundary conditions, completely specifies the eigenvalue problem for σ_1, given the primary shear flow $\bar{u}(z)$ and wave-field eigenfunction $\phi(z)$. This is much more complex than the corresponding problem (2.4) for weak mean flows, but some further progress is made in §§5 and 6 without recourse to numerical computation.

For *viscous* flows, the situation is still more complex. For weak $O(\epsilon^2)$ mean flow $\bar{u}(z)$, the viscous counterpart of (2.4) is discussed by Leibovich (1980). It is then necessary to suppose that the wave-slope parameter ϵ and 'wave Reynolds number' $R_{\mathrm{w}} \equiv \omega/\alpha^2\nu$ (ω being the wave frequency) are such that R_{w}^{-1} is $O(\epsilon^2)$: otherwise, viscous damping annihilates the $O(\epsilon^2)$ growth rate σ predicted by inviscid theory. For typical water waves, such viscous effects are too weak to suppress the instability. Immediate extension to $O(\epsilon)$ mean flows may be made, by similar arguments to those given in §2.3, on regarding R_{w}^{-1} to be $O(\epsilon^{\frac{3}{2}})$.

For $O(1)$ mean flows, the flow Reynolds number R must be taken as $O(\epsilon^{-1})$ to balance the potential $O(\epsilon)$ inviscid growth rate in this case. This replaces (2.9) by a fourth-order differential equation for $\hat{w}(z)$, and (3.6) by a second-order differential equation for $\hat{u}(z)$ in terms of $\hat{w}$. In contrast, viscous terms remain absent from (3.4) and (3.5) over most of the flow domain, but must be retained near walls and critical layers. Fortunately, linear theory remains valid at the latter locations, the role of nonlinearity in the critical layer only becoming significant at amplitudes ϵ that are $O(R^{-\frac{2}{3}})$ or more (see e.g. Benney & Bergeron 1969). Further progress for viscous flows will require a substantial computational effort.

4. Determination of $\hat{P}_1$ and $\hat{\phi}$

Without loss, the x-averaged flow field may be written as (cf. (2.7))

$$u = \bar{u}(z)+\delta\, e^{\sigma t}\cos ly\, U(z)+O(\delta^2, \epsilon\delta, \epsilon^2),$$
$$v = \epsilon\delta\, e^{\sigma t}\sin ly\, \hat{v}(z)+O(\epsilon^2\delta, \epsilon\delta^2),$$
$$w = \epsilon\delta\, e^{\sigma t}\cos ly\, \hat{w}(z)+O(\epsilon^2\delta, \epsilon\delta^2),$$

where σ is $O(\epsilon)$ and presumed real. Also, the x-periodic field has the form

$$u_1 = \epsilon\,\mathrm{Re}\,\{\phi'(z)\, e^{i\alpha x}\}+\epsilon\delta\,\mathrm{Re}\,\{\mathscr{U}_1(z)\cos ly\, e^{\sigma t}\, e^{i\alpha x}\}+O(\epsilon^2, \epsilon\delta^2),$$
$$u_2 = \epsilon\delta\,\mathrm{Re}\,\{\mathscr{U}_2(z)\sin ly\, e^{\sigma t}\, e^{i\alpha x}\}+O(\epsilon^2, \epsilon\delta^2),$$
$$u_3 = \epsilon\,\mathrm{Re}\,\{-i\alpha\phi(z)\, e^{i\alpha x}\}+\epsilon\delta\,\mathrm{Re}\,\{\mathscr{U}_3(z)\cos ly\, e^{\sigma t}\, e^{i\alpha x}\}+O(\epsilon^2, \epsilon\delta^2).$$

Here
$$l\hat{v}+\hat{w}' = 0, \quad i\alpha\mathscr{U}_1+l\mathscr{U}_2+\mathscr{U}_3' = 0,$$

to preserve continuity, the terms in $\mathscr{U}_j$ deriving from the modification of the $O(\epsilon)$ wave field by the $O(\delta)$ spanwise-periodic component of u.

The x-component of pseudomomentum may be written as

$$p_1 = \epsilon^2 P_1^0+\epsilon^2\delta\hat{P}_1 e^{\sigma t}\cos ly+O(\epsilon^4, \epsilon^3\delta, \epsilon^2\delta^2),$$

where P_1^0 is given by (3.1), and $\hat{P}_1$ may be recovered from (3.2). To do so, we write the particle displacements ξ_1, ξ_3 in the x- and z-directions as

$$\xi_j = \xi_j^0+\epsilon\delta\,\mathrm{Re}\,\{\Xi_j\cos ly\, e^{\sigma t}\, e^{i\alpha x}\} \quad (j = 1, 3),$$

where ξ_j^0 are just ξ_1 and ξ_3 as given in C (3.2) with $c_i = 0$. (The displacement component ξ_2, though non-zero at $O(\epsilon\delta)$, does not contribute to p_1 at the required order of approximation.)

It is readily found that

$$\Xi_1 = \left(\frac{-\phi U}{i\alpha\bar{u}^2}\right)' - \frac{i\alpha\bar{u}\mathscr{U}_1+\bar{u}'\mathscr{U}_3}{\alpha^2\bar{u}^2},$$

$$\Xi_3 = \frac{\phi U}{\bar{u}^2}+\frac{\mathscr{U}_3}{i\alpha\bar{u}},$$

where the prime denotes d/dz. Substitution in (3.2) and extraction of terms that are $O(\epsilon^2\delta)$ leads, after some reduction, to

$$2\hat{P}_1 = \left[\alpha^2\left|\frac{\phi}{\bar{u}}\right|^2+\left|\left(\frac{\phi}{\bar{u}}\right)'\right|^2-\frac{\bar{u}'}{\bar{u}}\left(\left|\frac{\phi}{\bar{u}}\right|^2\right)'\right]U+\left(\left|\frac{\phi}{\bar{u}}\right|^2\right)'U'$$
$$-\left\{\mathscr{U}_1\left(\frac{\phi^*}{\bar{u}}\right)'+\frac{\mathscr{U}_3}{i\alpha\bar{u}}\left[\bar{u}'\left(\frac{\phi^*}{\bar{u}}\right)'-\alpha^2\phi^*\right]+\mathrm{c.c.}\right\}. \quad (4.1)$$

On defining
$$\hat{\phi}(z) \equiv i\alpha^{-1}\mathscr{U}_3(z), \tag{4.2}$$
the continuity equation yields
$$\mathscr{U}_1 - \frac{il}{\alpha}\mathscr{U}_2 = \hat{\phi}'. \tag{4.3}$$

Also, the momentum equations yield
$$i\alpha(\bar{u}\mathscr{U}_1 + U\phi' - U'\phi) + \bar{u}'\mathscr{U}_3 = -i\alpha\mathscr{P}, \tag{4.4a}$$
$$i\alpha\bar{u}\mathscr{U}_2 = l\mathscr{P}, \tag{4.4b}$$
$$i\alpha\bar{u}\mathscr{U}_3 + \alpha^2 U\phi = -\mathscr{P}', \tag{4.4c}$$
where the $O(\epsilon\delta)$ pressure component is
$$\epsilon\delta\rho \operatorname{Re}\{\mathscr{P}(z)\cos ly\, e^{\sigma t}\, e^{i\alpha x}\}.$$
Elimination of $\mathscr{P}$ from (4.4a,b), gives
$$\mathscr{U}_1 + \frac{i\alpha}{l}\mathscr{U}_2 = \frac{\bar{u}'}{\bar{u}}\hat{\phi} - \bar{u}^{-1}(U\phi' - U'\phi),$$
which, together with (4.3), leads to
$$\mathscr{U}_1 = \frac{\alpha^2}{\alpha^2+l^2}\hat{\phi}' + \frac{l^2}{(\alpha^2+l^2)\,\bar{u}}(\bar{u}'\hat{\phi} - U\phi' + U'\phi).$$
On substituting for $\mathscr{U}_1$ and $\mathscr{U}_3$ in (4.1), one obtains
$$\hat{F}_1 = \mathscr{A}(z)\,U + \mathscr{B}(z)\,U' + \operatorname{Re}\{\mathscr{C}(z)\,\hat{\phi} + \mathscr{D}(z)\,\hat{\phi}'\} \tag{4.5}$$
after reduction, where
$$\mathscr{A}(z) = \frac{|\phi'|^2}{2\bar{u}^2}\frac{\alpha^2+3l^2}{\alpha^2+l^2} + \frac{|\phi|^2}{2\bar{u}^2}\left(\alpha^2 + \frac{3\bar{u}'^2}{\bar{u}^2}\right) - \frac{\bar{u}'(|\phi|^2)'}{2\bar{u}^3}\frac{2\alpha^2+3l^2}{\alpha^2+l^2},$$
$$\mathscr{B}(z) = \frac{\alpha^2}{2(\alpha^2+l^2)}\left(\frac{|\phi|^2}{\bar{u}^2}\right)',$$
$$\mathscr{C}(z) = -\frac{\alpha^2\phi^*}{\bar{u}} + \frac{\alpha^2\bar{u}'}{\bar{u}(\alpha^2+l^2)}\left(\frac{\phi^*}{\bar{u}}\right)',$$
$$\mathscr{D}(z) = \frac{-\alpha^2}{\alpha^2+l^2}\left(\frac{\phi^*}{\bar{u}}\right)'.$$
Also, from (3.6),
$$U = -\frac{\bar{u}'}{\sigma_1}\hat{w}. \tag{4.6}$$

Finally, elimination of $\mathscr{P}$, $\mathscr{U}_1$ and $\mathscr{U}_2$ from (4.4a–c) and (4.3) leads to the equation for $\hat{\phi}$:
$$\bar{u}\left[\frac{d^2}{dz^2} - (\alpha^2+l^2)\right]\hat{\phi} - \bar{u}''\hat{\phi} = -U\left[\frac{d^2}{dz^2} - (\alpha^2+l^2)\right]\phi + U''\phi, \tag{4.7}$$
where ϕ satisfies the Rayleigh equation (3.5). The corresponding equations for viscous flows may be obtained in a similar manner, that for $\hat{\phi}$ then being of fourth order. Note that, as an alternative to the above (Eulerian) formulation in terms of $\hat{\phi}$, the wave-field modification might equally well have been expressed in terms of the (Lagrangian) variable Ξ_3.

5. The approximation $l^2 \gg \alpha^2$

In an attempt to make further analytical progress, suppose that the wavelength $2\pi/\alpha$ of the fundamental waves is long compared with the characteristic lengthscale of the variations in $\bar{u}$: typically, the latter scale may be thought of as the channel width or boundary-layer thickness. Accordingly, we (meanwhile) consider that $\alpha^2 \ll 1$, while l^2 remains $O(1)$.

In this case, it is readily seen that

$$\mathscr{A}(z) = \frac{3}{2}\left[\frac{|\phi'|^2}{\bar{u}^2} + \frac{\bar{u}'^2|\phi|^2}{\bar{u}^4} - \frac{\bar{u}'(|\phi|^2)'}{\bar{u}^3}\right] + O(\alpha^2), \tag{5.1a}$$

$$\mathscr{B}(z),\ \mathscr{C}(z),\ \mathscr{D}(z) = O(\alpha^2). \tag{5.1b, c, d}$$

Therefore, to leading order, $\hat{P}_1$ is independent of the perturbed wave field $\hat{\phi}$, since (4.6) shows $\hat{\phi}$ to be $O(U\phi/\bar{u})$ in magnitude. (The case where a natural mode with wavenumber $(\alpha^2+l^2)^{\frac{1}{2}}$ has phase speed close to that with wavenumber α – i.e. near zero in the present frame – is the only possible exception: a possibility too remote to warrant examination here.)

Equation (2.9) then reduces to

$$\hat{w}'' + l^2[-1 + \sigma_1^{-2}\bar{u}'(P_{1,3}^0 - \bar{u}'\mathscr{A})]\,\hat{w} = O(\alpha^2\hat{w}),$$

where $\mathscr{A}$ denotes the above $O(1)$ approximation to $\mathscr{A}(z)$. However, from (3.1), it follows that $P_{1,3}^0 - \bar{u}'\mathscr{A}$ is also $O(\alpha^2)$. We therefore have $\hat{w}'' - l^2\hat{w} = O(\alpha^2\hat{w})$ when σ_1^2 is $O(1)$, and this admits only the trivial solution $\hat{w} = 0$ with boundary conditions $\hat{w} = 0$ at $z = z_1$ and z_2. In order to find non-trivial solutions, one must consider cases where σ_1 is $O(\alpha)$; but then the wave-field perturbation $\hat{\phi}$ cannot normally be neglected.

Fortunately, simplification remains possible for cases with

$$l^2 \gg 1, \quad \alpha = O(1),$$

which we now consider. When this is so,

$$\mathscr{A}(z) = \frac{3}{2}\left[\frac{|\phi'|^2 + \frac{1}{3}\alpha^2|\phi|^2}{\bar{u}^2} + \frac{\bar{u}'^2|\phi|^2}{\bar{u}^4} - \frac{\bar{u}'(|\phi|^2)'}{\bar{u}^3}\right] + O(l^{-2}), \tag{5.2a}$$

$$\mathscr{C}(z) = -\frac{\alpha^2\phi^*}{\bar{u}} + O(l^{-2}), \quad \mathscr{B}(z),\ \mathscr{D}(z) = O(l^{-2}). \tag{5.2b, c, d}$$

Also, since significant variations in $\hat{w}$, U and $\hat{\phi}$ now occur over lengthscales $O(l^{-1})$,

$$\hat{\phi} = O(\sigma_1^{-1}\hat{w}), \quad \hat{\phi}' = O(l\sigma_1^{-1}\hat{w}), \quad U' = O(l\sigma_1^{-1}\hat{w}).$$

Accordingly, (2.9) reduces to

$$l^{-2}\hat{w}'' + [\sigma_1^{-2}\bar{u}'(P_{1,3}^0 - \bar{u}'\mathscr{A}) - 1]\,\hat{w} = -\sigma_1^{-1}\bar{u}'\,\mathrm{Re}\,\{\mathscr{C}\hat{\phi}\} + O(l^{-1}\sigma_1^{-2}\hat{w}), \tag{5.3a}$$

or

$$\frac{d^2\hat{w}}{d(lz)^2} - \left[\frac{\bar{u}'\alpha^2}{\sigma_1^2}\left(\frac{|\phi|^2}{\bar{u}}\right)' + 1\right]\hat{w} = \frac{\bar{u}'\alpha^2}{\sigma_1\bar{u}}\,\mathrm{Re}\,\{\phi^*\hat{\phi}\} + O(l^{-1}\sigma_1^{-2}\hat{w}),$$

on substitution for $P_{1,3}^0$, $\mathscr{A}$ and $\mathscr{C}$.

Further, (4.7) simplifies to

$$\frac{d^2\hat{\phi}}{d(lz)^2}-\hat{\phi} = (l^{-2}U''+U)\frac{\phi}{\bar{u}}+O(l^{-2}) \tag{5.4a}$$

$$=\frac{-\bar{u}'\phi}{\sigma_1\bar{u}}\left[\frac{d^2\hat{w}}{d(lz)^2}+\hat{w}\right]+O(l^{-2}), \tag{5.4b}$$

on using the result (4.6) and recalling that $\bar{u}$, ϕ vary on lengthscales $O(1)$.

Since σ_1 is real, by assumption, the coupled equations (5.3*b*) and (5.4*b*) may be re-expressed in the form

$$\left[\frac{d^2}{d\zeta^2}-1+\sigma_1^{-2}G(z)\right]\hat{w} = r, \tag{5.5a}$$

$$\left[\frac{d^2}{d\zeta^2}-1\right]r = -\sigma_1^{-2}H(z)\left[\frac{d^2}{d\zeta^2}+1\right]\hat{w}, \tag{5.5b}$$

with $l^2 \to \infty$ and

$$\zeta \equiv lz, \quad G(z) \equiv -\alpha^2\bar{u}'\left(\frac{|\phi|^2}{\bar{u}}\right)', \tag{5.6a, b}$$

$$r \equiv \frac{\bar{u}'\alpha^2}{\sigma_1\bar{u}}\mathrm{Re}\,\{\phi^*\hat{\phi}\}, \quad H(z) \equiv \frac{\alpha^2\bar{u}'^2|\phi|^2}{\bar{u}^2}. \tag{5.6c, d}$$

Locally, $G(z)$ and $H(z)$ may be treated as constants for purposes of integration with respect to ζ.

These may be rewritten as a pair of uncoupled equations

$$\left[\frac{d^2}{d\zeta^2}-1-\lambda_k(z)\right]V_k = 0 \quad (k = 1,2), \tag{5.7}$$

where

$$\lambda_{1,2} \equiv \tfrac{1}{2}\{-\sigma_1^{-2}(G+H)\pm[\sigma_1^{-4}(G+H)^2-8\sigma_1^{-2}H]^{\frac{1}{2}}\}, \tag{5.8a}$$

$$V_k \equiv r+(\lambda_k+\sigma_1^{-2}H)\,\hat{w}. \tag{5.8b}$$

Since $\hat{w}$ and r must vanish on rigid boundaries at $z = z_1$ and z_2, V_k must also vanish there. Two linearly independent approximate solutions of (5.7), constructed by the WKB method, are

$$V_k \approx \text{const}\times(1+\lambda_k)^{-\frac{1}{4}}\exp\left\{\pm l\int^z[1+\lambda_k(s)]^{\frac{1}{2}}\,ds\right\} \quad (l^2 \gg 1).$$

Non-trivial solutions V_k satisfy the boundary conditions $V_k(z_1) = V_k(z_2) = 0$ if and only if

$$\int_{z_1}^{z_2}(1+\lambda_k)^{\frac{1}{2}}\,dz = \frac{N\pi i}{l} \tag{5.9}$$

for some integer N.

Clearly, when $1+\lambda_k$ is real and non-negative throughout the interval $[z_1, z_2]$, there is no non-trivial solution V_k. Conversely, solutions are sure to exist, for suitable values of N and σ_1, when $1+\lambda_k$ is real and negative throughout $[z_1, z_2]$. Equation (5.9) is then the (approximate) eigenvalue relation for the growth (or decay) rates σ_1 of a sequence of modes for the various values of N. A rather similar, though simpler, eigenvalue problem was thoroughly considered by Leibovich (1977*b*). In this, λ_k had the form $\sigma_1^{-2}F(z)$, where $F(z)$ was a monotonically increasing (or decreasing) function on a semi-infinite domain. Instability then arose whenever $F(z)$ was negative in a finite subinterval of the flow domain.

For flows with critical layers, the functions $G(z)$ and $H(z)$ are normally singular

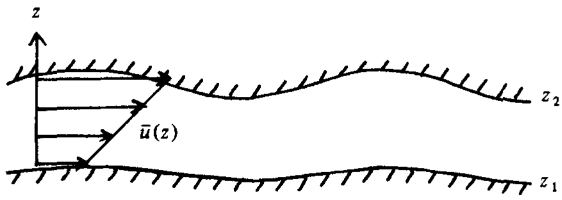

FIGURE 1. Example of wavy flows examined in §6.

where $\bar{u} = 0$. Treatment of such singularities requires care: in particular, it is not obvious that one is entitled to indent under these, in the complex z-plane, in evaluating the integral of (5.9). Fortunately, the instability under discussion does not rely on a critical-layer mechanism for its operation, and we therefore pass over such complexities. For simplicity, we now confine attention to a class of wavelike flows, without critical layers, which exhibit this instability.

6. A simple example: $\bar{u} = z$, $\alpha\phi = ae^{-\alpha z}$

We here consider a uniform shear flow $\bar{u} = z$ in some interval $z_1 \leqslant z \leqslant z_2$ that does not include the origin. Superposed on this flow is a two-dimensional wavelike disturbance with $\alpha\phi = a\exp(-\alpha z)$. This satisfies Rayleigh's equation (3.5) and we may suppose that the chosen form of ϕ may be realized by the imposition of suitable inhomogeneous boundary conditions at z_1 and z_2. The composite flow is a nearly uniform shear with wavelike perturbation, an example of which is sketched in figure 1.

Now,
$$G(z) = -a^2(z^{-1}e^{-2\alpha z})', \quad H(z) = a^2z^{-2}e^{-2\alpha z},$$
and hence
$$\lambda_{1,2} = q\{-1 \pm [1-2q^{-1}(1+\alpha z)^{-1}]^{\frac{1}{2}}\}, \tag{6.1}$$
where
$$q(z) \equiv \left(\frac{a}{\sigma_1}\right)^2 \frac{e^{-2\alpha z}(1+\alpha z)}{z^2}.$$
We restrict attention to a search for real eigenvalues σ_1, which would denote instability.

The functions $\lambda_{1,2}$ are real whenever
$$\left(\frac{\sigma_1}{\alpha a}\right)^2 < \tfrac{1}{2}e^{-2\alpha z}(1+\alpha z)^2(\alpha z)^{-2}. \tag{6.2}$$
Both satisfy $1+\lambda_k < 0$ when *either* $0 < \alpha z < 1$ and
$$2(\alpha z)^{-1}e^{-2\alpha z} < \left(\frac{\sigma_1}{\alpha a}\right)^2 < \tfrac{1}{2}e^{-2\alpha z}(1+\alpha z)^2(\alpha z)^{-2} \tag{6.3}$$
or $-1 < \alpha z < 0$. Also, $1+\lambda_1 > 0$ and $1+\lambda_2 < 0$ when $\alpha z > 0$ and
$$0 < \left(\frac{\sigma_1}{\alpha a}\right)^2 < \inf\left[e^{-2\alpha z}(1+\alpha z)(\alpha z)^{-2}, \frac{2}{\alpha z}e^{-2\alpha z}\right], \tag{6.4}$$
the right-hand limit being $(2/\alpha z)e^{-2\alpha z}$ whenever $0 < \alpha z < 1$. Both values λ_k satisfy $1+\lambda_k > 0$ when the conditions (6.3) are satisfied together with $\alpha z > 1$; also when the conditions (6.2) and $\alpha z < -1$ are met.

When *both* $1+\lambda_1$ and $1+\lambda_2$ remain positive throughout the flow domain $[z_1, z_2]$, no solution of (5.9) exists.

When $1+\lambda_1 > 0$ but $1+\lambda_2 < 0$ throughout the flow, there must exist a discrete set of real eigenvalues σ_1 that satisfy (5.9) with $k = 2$. Since l is assumed large, we may normally expect these eigenvalues to be associated with rather large (positive or negative) integers N. When $0 < \alpha z < 1$ throughout $[z_1, z_2]$, these eigenvalues lie in the range (6.4): the infimum then refers to the smallest value attained in the flow domain. In contrast, (5.9) has no solutions when $k = 1$: the trivial solution $V_1 = 0$ is then the only one available. Since there are non-trivial solutions for V_2, the corresponding solutions for $\hat{w}$ and r are both non-trivial.

When both $1+\lambda_1$ and $1+\lambda_2$ are negative throughout $[z_1, z_2]$, there are three possibilities: (i) there are eigenvalues σ_1 of (5.9) with $k = 1$, for some values $N = N_1$, that are not eigenvalues of (5.9) with $k = 2$ for any integer N; (ii) there are eigenvalues σ_1 of (5.9) with $k = 2$, for values $N = N_2$, that are not eigenvalues of (5.9) with $k = 1$ for any integer N; (iii) there are eigenvalues σ_1 that satisfy (5.9), with both $k = 1$ and 2, for some pairs of integers N_1, N_2. In cases (i) and (ii) V_2 and V_1 respectively are identically zero while the other has non-trivial solutions; in case (iii) both V_1 and V_2 have non-trivial solutions.

In all the above cases for which real eigenvalues σ_1 occur, there exist exponentially growing disturbances, since real roots appear in pairs, with opposite signs. The flow is then unstable to disturbances of the assumed form, in the absence of viscosity. In particular, such instability occurs whenever $\alpha z > 0$ or $-1 < \alpha z < 0$ throughout the flow, but not when $\alpha z < -1$ throughout the flow. The former instability criterion, that $\alpha z > 0$ throughout the flow, is satisfied whenever the local wave amplitude $|\phi|$ *everywhere decreases in the direction of increasing speed* of the primary flow $\bar{u}$ relative to the wave. When the wave amplitude $|\phi|$ everywhere *increases* in the direction of increasing speed $|\bar{u}|$ (as for $\alpha > 0$, $z < 0$), the instability persists when the wave is sufficiently long that $|\alpha z| < 1$ throughout $[z_1, z_2]$; but no such instability is evident for waves short enough that $|\alpha z| > 1$ everywhere in $[z_1, z_2]$.

Cases where $1+\lambda_k$ change sign within the flow domain or where λ_k become complex have not been considered. Detailed examination of the eigenvalue relations (5.9) would then be necessary, to determine whether growing modes are permitted (cf. Leibovich 1977*b*). We here rest content with having established the existence of instability for the above, simpler examples.

7. Physical discussion

The underlying physical mechanism for instability is not immediately apparent from the foregoing analysis: indeed, the mechanism is a rather subtle one.

In the case of weak shear flows $O(\epsilon^2)$, Craik (1977) and Leibovich (1977*b*) have given a clear account of the physical processes. Craik's interpretation of the instability is a *kinematical* one (with the 'dynamics' contained in Helmholtz's vorticity laws). An initially unidirectional mean shear flow is envisaged as having weak spanwise-periodic variations in magnitude. Associated with these variations is a spanwise-periodic vertical vorticity, which is necessarily tilted by the vertical gradient of the Stokes drift, since vortex lines and particle paths must coincide in inviscid flows. The resultant x-vorticity induces spanwise-periodic vertical motions, which further distort the mean shear flow. This intensifies the initially postulated variations, provided that the Stokes drift gradient and mean shear have the same sign. This 'feedback cycle' leads to exponential growth of the spanwise-periodic disturbances. Leibovich's (1977*b*) interpretation in terms of a 'vortex force' $\mathbf{u}_1^S \times \boldsymbol{\omega}$ is essentially equivalent to the above.

The presence of waves is crucial for the mechanism to work: for it is the difference between the Lagrangian-mean and Eulerian-mean flows that generates the longitudinal vorticity. Tilting of vertical vorticity by the Eulerian-mean shear $\bar{u}'(z)$ is exactly offset by the tilting of mean spanwise vorticity by the spanwise-periodic component of the x-velocity.

For such weak mean flows, we now give an alternative and complementary physical interpretation, from a purely Eulerian viewpoint that does not explicitly involve the Lagrangian concept of Stokes drift. (The author is grateful to Professor G. K. Batchelor and Dr M. E. McIntyre for indicating that such an Eulerian interpretation must exist.) The underlying mechanism is a variant of the familiar Taylor–Görtler instability of curved flows, in which the curvature derives from the undulatory streamlines. In the inviscid approximation, the governing equation of Taylor–Görtler instability has the form (cf. Lin 1955, p. 97)

$$\hat{w}_{,zz}+l^2\left[\frac{2K\bar{u}\bar{u}'}{\sigma^2}-1\right]\hat{w}=0,$$

where K denotes the curvature of streamlines. This bears an obvious resemblance to the result (2.4) for weak mean shear flows and also to the result (5.3) for strong shear flows (though a further term then appears on the right-hand side). In the former case, $2K\bar{u}$ is replaced by the Stokes drift gradient $\bar{u}_1^{S\prime}$ and in the latter by $P_1^{0\prime}-\bar{u}'\mathscr{A}$.

Now, the curvature of a particular undulatory streamline is $-\alpha^2\xi_3$ to leading order, and the local tangential velocity along this streamline is just $u=\bar{u}+\hat{u}_1$, correct to $o(\epsilon)$, where ξ_3 and $\hat{u}_1$ are given by C(3.2). Proceeding heuristically, we construct the x-average at a fixed level z:

$$2\overline{Ku}=-2\alpha^2\overline{\xi_3(\bar{u}+\hat{u}_1)}=-\frac{\epsilon^2\alpha^2}{2\bar{u}}(|\phi|^2)'. \tag{7.1}$$

But, for weak mean shear flows, $\bar{u}$ is just $-c$ to leading order, and the Stokes drift C(3.3) reduces to

$$\bar{u}_1^S=\tfrac{1}{4}\epsilon^2c^{-1}(|\phi|^2)''+O(\epsilon^3),$$

where $\phi''=\alpha^2\phi$. Accordingly,

$$\bar{u}_1^{S\prime}=\epsilon^2\alpha^2c^{-1}(|\phi|^2)',\quad 2\overline{Ku}=\tfrac{1}{2}\epsilon^2\alpha^2c^{-1}(|\phi|^2)',$$

which differ only by the factor of $\frac{1}{2}$.

In fact, it is necessary to evaluate the curvature K correctly to $O(\epsilon^2)$, and this requires some care. The streamline passing through a point (x_0,z_0) is

$$z+\epsilon\,\mathrm{Re}\left\{\frac{-\phi(z)}{\bar{u}(z)}e^{i\alpha x}\right\}=z_0+\epsilon\,\mathrm{Re}\left\{\frac{-\phi(z_0)}{\bar{u}(z_0)}e^{i\alpha x_0}\right\}+O(\epsilon^2),$$

where all the omitted $O(\epsilon^2)$ terms are second harmonics in $\exp(2i\alpha x)$ or $\exp(2i\alpha x_0)$ that play no essential part in the subsequent analysis. This may be rewritten as

$$z-z_0=\epsilon\,\mathrm{Re}\left\{\frac{\phi(z_0)}{\bar{u}(z_0)}(e^{i\alpha x}-e^{i\alpha x_0})\right\}\left[1-\epsilon\,\mathrm{Re}\left\{\left(\frac{\phi(z)}{\bar{u}(z)}\right)'_{z=z_0}e^{i\alpha x}\right\}\right]+\text{second harmonics},$$

and the curvature K at (x_0,z_0) of this streamline is

$$K(x_0,z_0)=\left(\frac{\partial^2z}{\partial x^2}\right)_{\substack{x=x_0\\ z=z_0}}=\epsilon\,\mathrm{Re}\left\{\frac{-\alpha^2\phi(z_0)}{\bar{u}(z_0)}e^{i\alpha x_0}\right\}-\frac{\epsilon^2\alpha^2}{4}\left(\left|\frac{\phi(z)}{\bar{u}(z)}\right|^2\right)'_{z=z_0}+\text{second harmonics}.$$

With weak mean shear flows, $\bar{u} = -c$ and the $O(\epsilon^2)$ mean correction to $K(x_0, z_0)$ is $-\frac{1}{4}\epsilon^2\alpha^2 c^{-2}(|\phi|^2)'$, which yields an additional term to $2\overline{Ku}$ that is equal to that found above. That is to say, $2\overline{Ku} = \bar{u}_1^{S\prime}$ as required; and the instability mechanism with weak shear is *precisely* a Taylor–Görtler instability, but averaged in the x-direction, with the curvature of the flow deriving from the undulatory streamlines.

Of course, this pleasing physical interpretation in no way contradicts the kinematical interpretation in terms of vortex-line deformation: the two are complementary, as is also the case for the familiar Taylor–Görtler instability. Nor should the Eulerian analysis sketched above be viewed as anything more than a heuristic demonstration: a rigorous Eulerian analysis certainly exceeds the GLM derivation in its complexity.

For strong $O(1)$ shear flows, the instability mechanism is no longer precisely equivalent to the (averaged) Taylor–Görtler instability, but obvious similarities remain. The inviscid eigenvalue problem defined by (5.7), (5.8) and boundary conditions resembles that for Couette flow (cf. Drazin & Reid 1981, p. 83), but with more-complicated λ_k. Also, the result of §6, that instability occurs when the wave amplitude everywhere increases in the direction of decreasing speed (relative to the stationary wave profile), is in line with the results of the Craik–Leibovich theory for weak shear flows. That this instability persists, in the example of §6, when $-1 < \alpha z < 0$ would seem to suggest that the additional terms of the strong shear-flow problem tend to enhance, rather than inhibit, the instability.

8. Conclusion

An instability mechanism has been revealed that may lead to the exponential growth of longitudinal-vortex modes in strong shear flows subjected to two-dimensional wavelike disturbances. The wave field is initially independent of the spanwise coordinate, and the mechanism is unrelated to that first proposed by Benney & Lin (1960) or to any other theoretical model that relies crucially on forcing by pre-existing oblique-wave modes.

In effect, the analysis is a generalization to strong shear flows of that developed by Craik (1977) and Leibovich (1977*b*) for the growth of Langmuir circulations. This largely employs the generalized Lagrangian-mean formulation of Andrews & McIntyre (1978*a*, *b*), which, though yielding the governing equations less readily than in cases of weak mean shear, remains preferable to a purely Eulerian analysis.

With weak $O(\epsilon^2)$ or $O(\epsilon)$ mean flows, the GLM equations (2.1) are effectively complete, since the back-effect of mean-flow evolution on the wave field is negligible. They then reduce directly to their Eulerian counterparts, since Stokes drift and pseudomomentum are equal at $O(\epsilon^2)$. Considerable saving of effort and benefit of physical insight then accrue: this was shown in the case of Craik–Leibovich instability, and it is doubtless so in other cases also. With strong mean shear flows, the GLM equations may still reduce to their Eulerian-mean counterparts with some savings, but these equations are no longer complete and a separate calculation of the modifications to the wave field and associated pseudomomentum becomes necessary.

The physical interpretation of the instability for weak shear was further elucidated. The mechanism is an inviscid one, which may be viewed in terms of distortion of mean vortex lines by the Stokes drift of the wave field, or in terms of an equivalent 'vortex force', as already described by Craik (1977) and Leibovich (1977*b*). Alternatively, it may be seen as a Taylor–Görtler instability of curved flows, viewed in a reference frame in which the waves are brought to rest and averaged in the downstream direction, the local curvature being that of the undulating streamlines.

For strong shear flows, the physical mechanism cannot be completely understood in such simple terms, but similarities with the weak-shear instability remain. The linear eigenvalue problem that governs the stability of the longitudinal vortices is then fairly complex, and will normally require numerical solution for particular flows. But, in the inviscid limit with small spanwise spacing of the vortices, the problem simplifies sufficiently for definite results to be obtained analytically. The existence of the instability mechanism with strong shear was demonstrated by an example, and it may be expected to operate in a wide class of similar flows. Extension of the analysis to viscous flows and all spanwise wavenumbers is reasonably straightforward in principle, but would entail considerable computational effort.

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